Nghia Ho

Where boredom, free time, and curiosity meet together

Loopy belief propagation, Markov Random Field, stereo vision

In this tutorial I'll be discussing how to use Markov Random Fields and Loopy Belief Propagation to solve for the stereo problem. I picked stereo vision because it seemed like a good example to begin with, but the technique is general and can be adapted to other vision problems easily.

I try my best to make this topic as easy to understand as possible. Most resources on this topic are very heavy on the maths side, which makes it hard for those who aren't maths buff to grasp. I on the other hand will try to keep the maths to the bare minimum.

If you have any suggestions for improvement please leave a comment!

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Stereo vision problem

The stereo problem asks given a stereo image pair, such as the one below, how can we recover the depth information. I've chosen the popular Tsukuba stereo pair commonly used by academics to benchmark stereo algorithms.

— Left image

— Right image

The images are taken at slightly different view, similar to our eyes. We know from the parallax effect that objects closer to us will appear to move quicker than those further away. The same idea applies here. We expect the pixels on the statue to have a larger disparity than those in the background.

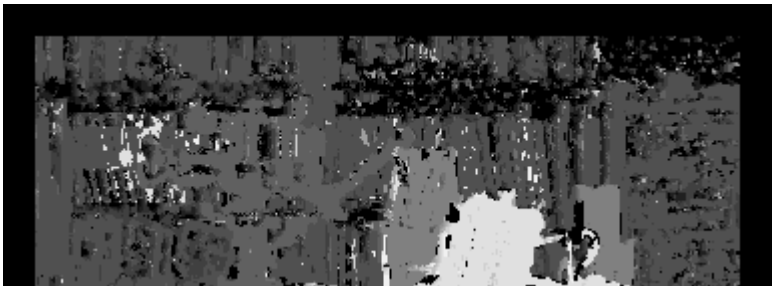
The Tsukuba stereo pairs have been rectified such that each pixel row in the left image is perfectly corresponds to the right. Or in multi-view geometry speak, the scan lines are the epipolar lines. What this means is that a pixel in the left image has a matching correspondence in the right image somewhere along the same row (assuming it is not occluded). This greatly simplifies the problem because pixel matching just becomes a 1D horizontal line search. This is probably the simplest stereo vision setup you can work with.

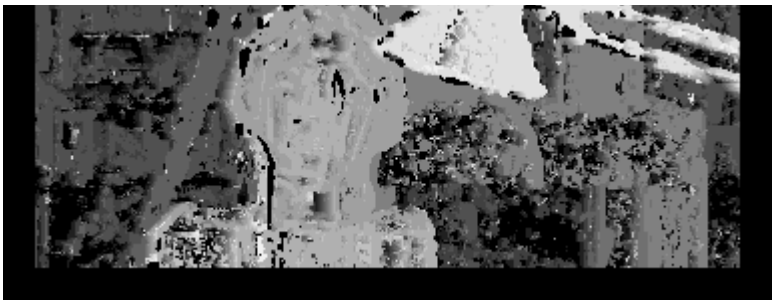
Naive attempt at recovering disparity

Having explained the stereo problem, lets attempt to recover the disparity map using simple block matching with the following parameters:

- converted image to greyscale
- 16 disparity levels (pixel search range)
- 5×5 block
- sum of absolute difference (SAD) scoring
- 16 pixel border

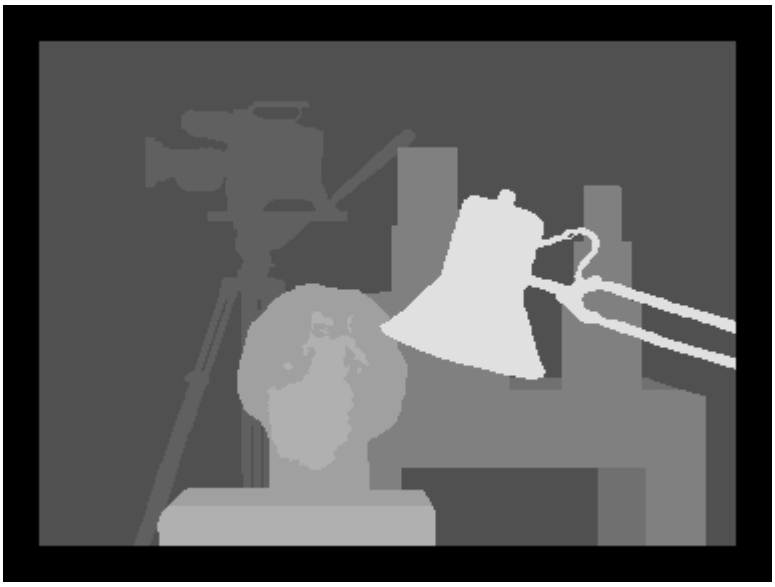
This produces the following disparity map





— Disparity map from matching using 5×5 block

As you can see, the disparity map is rather noisy with holes here and there. Kind of ugly really. We can make out the statue, lamp, and maybe some of the background, with pixels closer to us being brighter. Compare this with the ground truth disparity map.



— Ground truth disparity map

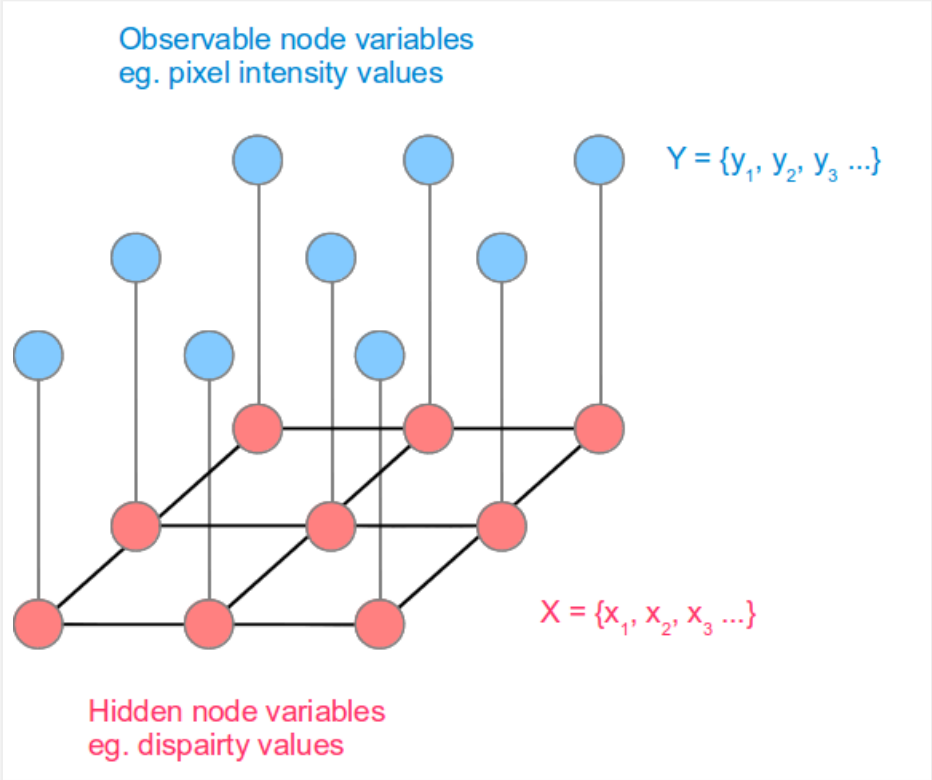
We could improve the results with some post filtering …. or employ some fancy Markov Random Field (MRF). Lets choose the fancy route!

Markov Random Field

The problem with recovering the disparity map just by looking at each individual pixel and finding the best match is that it ignores neighbouring/context/spatial information. We expect pixels near each other to have similar disparity, unless there is a genuine boundary. This is where MRF are useful.

MRF are undirected graphical models that can encode spatial dependencies. Like all graphical models,

they consist of nodes and links. However, unlike some graphical model eg. Bayesian network, they can contain cycles/loops. The stereo problem can be modeled using an MRF as follows for a 3×3 image.



— MRF for a 3×3 image

The blue nodes are the ***observed*** variables, which in our case are the pixel intensity values. The pink nodes are the ***hidden*** variables, which represents the disparity values we are trying to find. The hidden variable values are more generally referred to as ***labels***. I'll use the term 'labels' here on for generality.

The links between each node represents a dependency. So for example, the centre pixel's hidden node depends ONLY on its 4 neighbouring hidden nodes + the observed node. This rather strong assumption that a node's state depends only on its immediate neighours is called a Markov assumption. The beauty of this simple assumption is that it allows us to solve for the hidden variables in a reasonably efficient manner.

MRF formulation

We can formulate the stereo problem in terms of the MRF as the following energy function

$$energy(Y,X) = \sum_i DataCost(y_i, x_i) + \sum_{j=\text{neighbours of i}} SmoothnessCost(x_i, x_j)$$

The variables Y and X are the observed and hidden node respectively, i is the pixel index, j are the neighbouring nodes of node x_i. Refer to the MRF diagram above.

The energy function basically sums up all the cost at each link given an image Y and some labeling X. The aim is to find a labeling for X (disparity map for stereo) that produces the lowest energy. The energy function contains two functions that we will now look at, DataCost and SmoothnessCost.

DataCost

The DataCost function, or sometimes referred to as the **unary** energy/term/potential, returns the cost/penalty of assigning a label value of x_i to data y_i. This means we want a low cost for good matches and high value otherwise. An obvious choice is the sum of absolute difference mentioned earlier. There's also sum of square difference (SSD) and a whole bunch of others out there.

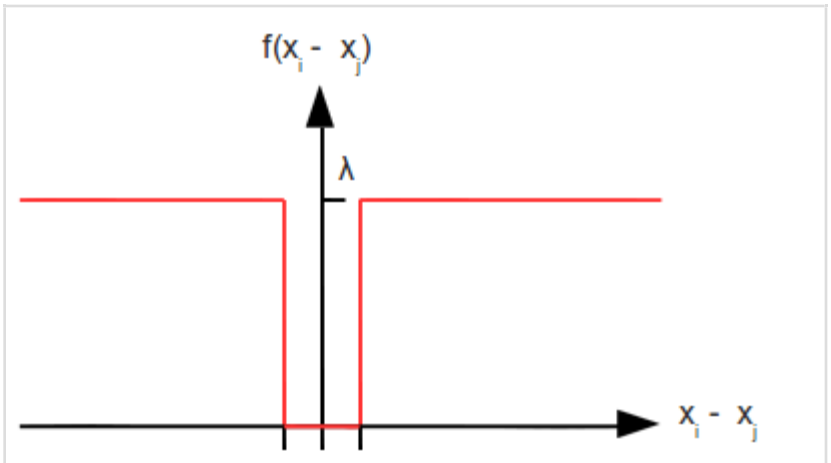
Here's an example of a very simple DataCost function that only compares a single pixel (in practice you'd calculate it over a block). The pseudo code is:

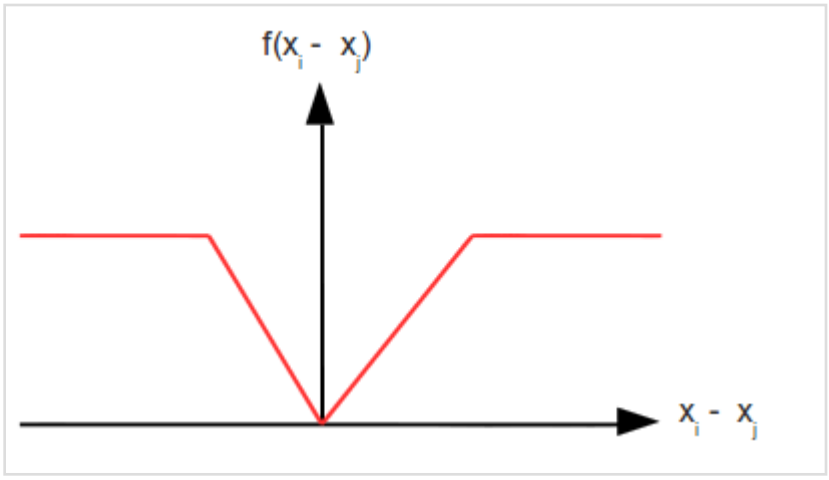
```
function DataCost(i, label)
    y = (int)(i / imageWidth) /* integer round down */
    x = i - y*imageWidth

    d = abs(leftImage(x,y) - rightImage(x - label),y)

    return d
end
```

The disparity direction taken on the rightImage is of course dependent on the stereo pair you are given.

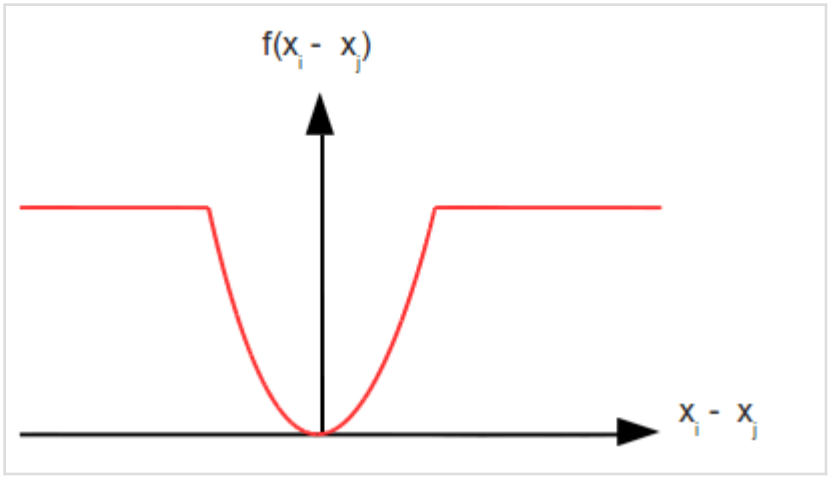
SmoothnessCost

The SmoothnessCost function, or sometimes referred to as the **pairwise** energy/term/potential, enforces smooth labeling across adjacent hidden nodes. To do this we need a function that penalises adjacent labels that are different. Below is a table showing some commonly used cost functions.





$$f(n) = \lambda \times \min(|n|, K)$$
Truncated linear model.



$$f(n) = \lambda \times \min(n^2, K)$$
Truncated quadratic model.

The Potts model is a binary penalising function with a single tunable λ variable. This value controls how much smoothing is applied. The linear and quadratic models have an extra parameter K. K is a truncation value that caps the maximum penalty.

Choosing a suitable DataCost and Smoothness function as well as the parameter seems like a black art, at least to me. The papers I've come across don't talk about how they've chosen their parameters. My guess is through experimentation.

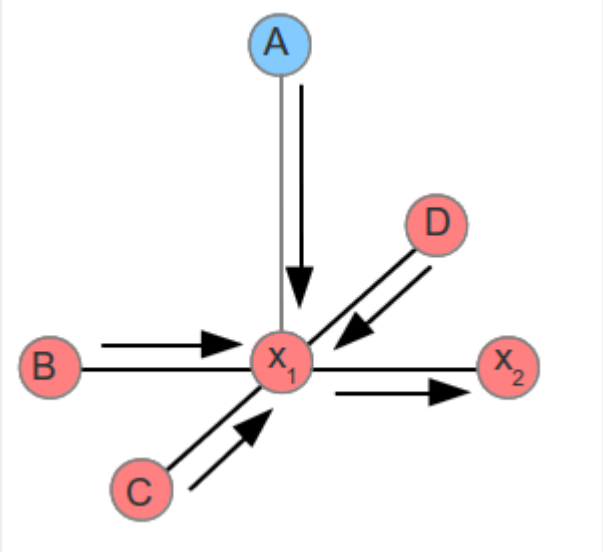
Loopy Belief Propagation

So we've got the MRF formulation patted down. Lets imagine we've gone ahead and chosen a DataCost

and SmoothnesCost function as well as some parameters to experiment with. But exactly how do we solve the for the energy function? Can we brute force this and try all combinations? Lets see, for the Tsukuba image we've got 384×288 = 110592 pixel and 16 disparity levels, which gives us 16^110592 combinations … hmmm where's my quantum computer. So finding an *exact* solution is a no go, but if we can settle for an approximate solution then we're in luck. The loopy belief propagation (LBP) algorithm is one of many algorithms (Graph cut, ICM …) that can find an approximate solution for a MRF.

The original belief propagation algorithm was proposed by Pearl in 1988 for finding exact marginals on trees. Trees are graphs that contain no loops. It turns out the same algorithm can be applied to general graphs, those that can contain loops, hence the 'loopy' in the name. However there is no guarantee of convergence.

LBP is a message passing algorithm. A node passes a message to an adjacent node only when it has received all incoming messages, excluding the message from the destination node to itself. Below shows an example of a message being passed from x_1 to x_2.



— Message passing

Node x_1 waits for messages from nodes A,B,C,D before sending its message to x_2. As a reminder, it **does not** send the message from $x_2 \rightarrow x_1$ back to x_2.

Lets define the message formally as:

$$msg_{i \rightarrow j}(l)$$

This is read as node i sends a message to node j about label l. It is node i's belief about node j regarding label l. **Message passing is only performed on the hidden nodes**.

A complete message includes all possible labels. For example you can think of node i's message to node j along the lines of:

“hey node j, I believe you have label 0 with cost/probability s0″

“hey node j, I believe you have label 1 with cost/probability s1″

…

“hey node j, I believe you have label N with cost/probability sn”

Node i maintains all possible beliefs about node j. The choice of using cost/penalty or probabilities is dependent on the choice of the MRF energy formulation.

LBP Algorithm

Putting together what has been discussed so far, below is one possible implementation of the LBP algorithm for computer vision.

```
function LoopyBeliefPropgation
    initialise all messages

    for t iterations
        at every pixel pass messages right
        at every pixel pass messages left
        at every pixel pass messages up
        at every pixel pass messages down
    end for

    find the best label at every pixel i by calculating belief
end
```

The first step is an initialisation of the messages. Earlier on I mentioned that a node has to wait for all incoming messages before sending it off. This causes a bit of a chicken and egg problem on a graph with loops because technically every node would be waiting forever and nothing would get sent. To get around this we initialse all the messages to some constant, giving all the nodes the incoming messages they need to proceed. The initialisation constant is usually either 0 or 1 depending on the energy formuation chosen.

The main part of LBP is iterative. As with most iterative algorithms we can choose to run it for a fixed number of iterations or terminate when the change in energy drops below a threshold, or any other suitable criteria.

At each iteration messages are passed around the MRF. The choice of the message passing scheme is arbitrary. In this example I’ve chosen right, left, up down, but any other sequence is valid. Different sequences will generally produce different results.

Once LBP iteration completes we can find the best label at every pixel by calculating its belief.

We will now look at the three main parts of the LBP algorithm in detail:

1. **message update**
2. **message initialisation**
3. **belief**

for three different algorithms:

1. **sum-product**
2. **max-product**
3. **min-sum**

Each of these algorithms will directly minimise the energy function presented earlier on.

Sum-Product – message update

The sum-product is usually the first algorithm people are taught in regards to belief propagation. It is formulated as:

$$msg_{i \to j}(l) = \sum_{l' \in \text{all labels}} \left[\begin{array}{c} exp(-DataCost(y_i, l)) \, exp(-SmoothnessCost(l, l')) \times \\ \prod_{k = \left(\begin{array}{c} \text{neighbours of i} \\ \text{except j} \end{array} \right)} msg_{k \to i}(l') \end{array} \right]$$

I've written the equation slightly more verbose than what you would typically see in an academic paper. It is split across two lines. As a reminder

- $msg_{i \to j}(l)$ = message from node i to node j for label l
- y_i = pixel intensity observed at pixel i

$\sum_{l' \in \text{all labels}}$ means loops over all possible labels (eg. 0 to 15 disparity labels).

This equation is called sum-product because of the outer summation and inner product.

The sum-product algorithm is designed to operate on probability quantities. The exp() function converts the DataCost/SmoothnessCost penalty function into a valid probability value between 0 and , where 0 is "bad" and 1 is "good".

The inner products (terms inside the outer summation) is the **joint probability** of DataCost, SmoothnessCost and all the incoming messages for a given label l'. While the outer summation is a **marginalisation** over the variable l'.

Remember that the complete message is a vector of messages eg.

$$msg_{i \to j} = \begin{bmatrix} msg_{i \to j}(0) \\ msg_{i \to j}(1) \\ msg_{i \to j}(2) \\ \dots \end{bmatrix}$$

Just to clarify, when I write $msg_{i \to j}$ **without the indexing** it means the complete message vector.

You might have noticed that for every label l we have to do a summation over l'. This operation is quadratic in complexity O(L^2), two for loops basically. This is important to keep in mind because increasing the number of labels will incur a quadratic penalty.

Sum-Product – normalisation

One issue arising from continuously multiplying probabilities is that it will tend towards zero and eventually hit floating point limits. To avoid this, the complete message vector needs to be normalised before sending:

$$msg_{i \to j} = \frac{msg_{i \to j}}{\sum_l msg_{i \to j}(l)}$$

Sum-Product – initialisation

Since we are dealing with probabilities all the nodes have their messages initialised to 1 before running LBP.

Sum-Product – belief

The belief at any given node is a product of all incoming messages:

$$Belief(x_i = l) = exp(-DataCost(y_i, l)) \times \prod_{k=\text{neighbours of i}} msg_{k \to i}(l)$$

This is read as the belief that node i takes on label l. To find the best label you would go through all possible labels and see which one has the highest belief.

Max-Product – message update

The sum-product finds the best label individually at each node. But this may not actually be the best labeling as a whole! It might sound a bit counter intuitive, but here is a simple example using two binary variables x,y that illustrates this. Lets say x,y has the following probability table:

P(x,y)	x=0	x=1	
y=0	0.5	0.4	P(y=0) = 0.9
y=1	0.1	0.3	P(y=1) = 0.4
	P(x=0) = 0.6	P(x=1) = 0.7	

The marginals for each variable are on the outer edge of the table. If we were to pick the best configuration using only the individual marginals then we would pick x=1 (0.7) and y=0 (0.9), giving a probability of P(x=1, y=0) = 0.4. But the best configuration is when P(x=0, y=0) = 0.5. What we are really interested in is the max joint probability. This type of problem arises a lot in ***maximum a posteriori*** **(MAP) assignment** problems, where we want to find the **BEST** assignment as a whole. The max-product algorithm addresses this issue with a slight modification to the sum-product message update equation:

$$msg_{i \to j}(l) = \max_{l' \in \text{all labels}} \left[\begin{array}{c} exp\left(-DataCost\left(y_i, l\right)\right) exp\left(-SmoothnessCost\left(l, l'\right)\right) \times \\ \prod_{k=\left(\begin{array}{c} \text{neighours of i} \\ \text{except j} \end{array} \right)} msg_{k \to i}\left(l'\right) \end{array} \right]$$

So instead of summing over all possible labels l' (marginalisation) it keeps track of the largest marginal probability.

The initialisation, normalisation and belief calculation remains the same. The belief at each node is the **max-marginal**.

One important assumption of the algorithm is that the max-marginal values are unique at each node. When there are ties then it can complicate things. You can break ties by picking one randomly but this will no longer guarantee a MAP assignment. Or you can resolve systematically using something call *backtracking*, which I won't dive into because I'm not familiar with it.

Min-Sum – message update

The min-sum algorithm is similar to max-product, in that it finds the max-marginals at each node, but operates in log-space. The min-sum update is given as:

$$msg_{i \to j}(l) = \min_{l' \in \text{all labels}} \left[\begin{array}{c} DataCost\left(y_i, l\right) + SmoothnessCost\left(l, l'\right) + \\ \sum_{k=\text{neighours of i except j}} msg_{k \to i}\left(l'\right) \end{array} \right]$$

Min-sum is a minimisation problem, because we are trying to find the lowest cost. If we include minus signs on DataCost and SmoothnessCost then it would be a max-sum.

Min-Sum -initialisation

All the messages are initialised to 0.

Min-Sum – normalisation

Normalisation is less straight forward in log-space, being

$$A = log \sum_{l} exp\left(msg_{i \to j}\left(l\right)\right)$$

$$msg_{i \to j} = msg_{i \to j} - A$$

In practice there's a good chance you can skip the nornamlisation step. Min-sum operates in log-space so it's harder to reach underflow/overflow limits, since it's only doing additions. This is true if the DataCost and Smoothness function returns values that aren't too large.

Min-Sum – belief

The belief for min-sum is

$$Belief\left(x_i = l\right) = DataCost\left(y_i, l\right) + \sum_{k=neighbours of x_i} msg_{k \to i}\left(l\right)$$

The term *belief* here might be a bit misleading. We are actually looking for the belief with the smallest value, a better name might simply be *cost* to avoid confusion.

Implementation consideration

If possible, use the min-sum algorithm because it's the most computationally efficient out of the three algorithms mentioned. It doesn't have any expensive exp() functions and uses mainly addition operations. It is also easy to implement using only integer data types, which may give some performance boost.

If you need to implement the sum-product for whatever reason then you will probably need to introduce a scaling constant when calculating exp() to avoid underflow eg. exp(-DataCost(…)*scaling) * exp(-SmoothnessCost(..)*scaling), where scaling is a value between 0 and 1.

Stereo results

Alright, we've gone through a fair bit of theory but now lets revisit the stereo vision problem! We'll use the Tsukuba images again and run belief propagation with the following parameters:

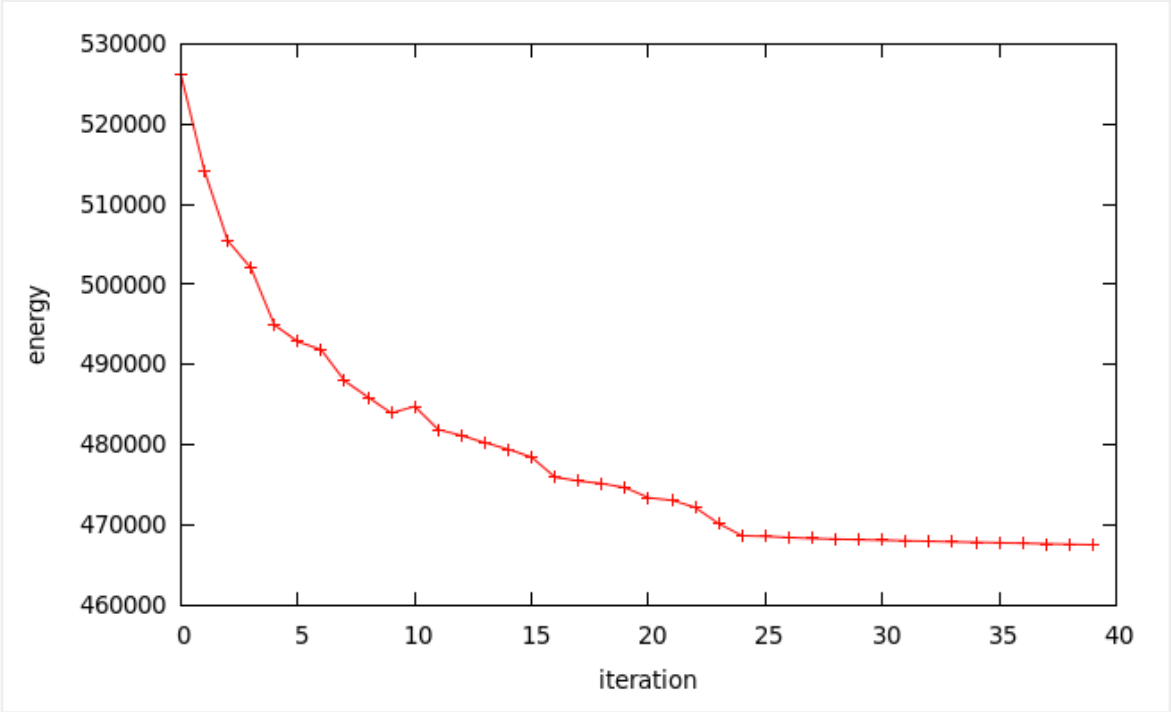
- converted image to greyscale
- 5×5 block
- 16 disparity labels (ranging from 0 to 15)
- DataCost using **linear model**
- SmoothnessCost using **truncated linear model**, truncated at 2, with $\lambda = 20$
- Min-sum optimisation algorithm
- 40 iterations of loopy belief propagation

This gives us the following disparity map

— Disparity map using loopy belief propagation

which is a much nicer disparity map than on our first attempt! The disparities are much smoother. We can make out the individual objects much better than before, especially the camera on the tripod in the background. Below shows a plot of the energy for each iteration.

There's a slight increase in energy on the 10th iteration, but after that it monotonically decreases where it flattens off around the 25th iteration. Remember that the LBP algorithm is not guarantee to converge, which means there's no guarantee that the energy will decrease at every iteration either.



— LBP convergence graph

What's interesting to compare is the energy obtained from LBP and that from the ground truth image. Using

the same energy function for generating the disparity map, the Tsukuba ground truth returns an energy of 717,701. This is actually higher than the final energy returned from LBP. So what can we say from this? This suggests our energy formulation doesn't quite accurately model real life, in our case its quite an over simplification. The more recent algorithms take occlusion, and possibly more, into account. Have a look at http://vision.middlebury.edu/stereo for the latest and greatest in stereo algorithms.

Download

You can download my implementation of LBP here. It'll generate the same results you see on this site.

LoopyBP.tar.bz2

It requires OpenCV 2.x installed.

On Linux, type **make** to compile and run it via **bin/Release/LoopyBP**. It'll read the two Tsukuba image files and output the disparity image as **output.png**. I've also included the ground truth image for reference.

License

The code is released under the simplified BSD license, see LICENSE.txt inside the package for more information.

Useful resources

Here are some resources that I found useful in creating this tutorial.

- General background knowledge on probabilistic graphical models was learnt from Coursera's PGM class. This was a very challenging but rewarding course. https://www.coursera.org/course/pgm
- Diagram showing basic MRF model for image processing with some general description, http://classes.soe.ucsc.edu/cmps290c/Spring04/proj/BPApp.pdf
- Parameters for the Tsukuba pair were obtained from http://vision.middlebury.edu/MRF/results/tsukuba/index.html
- Good and concise reference on belief propagation, https://github.com/daviddoria/Tutorials/blob/master/BeliefPropagation/BeliefPropagation.pdf?raw=true
- Slides explaining subtleness of max-product vs sum-product, http://www.cedar.buffalo.edu/~srihari/CSE574/Chap8/Ch8-GraphicalModelInference/Ch8.3.3-Max-SumAlg.pdf

19 THOUGHTS ON "LOOPY BELIEF PROPAGATION, MARKOV RANDOM FIELD, STEREO VISION"

Javier

on **July 10, 2012 at 4:38 pm** said:

Great tutorial, thanks!

nghiaho12 on **July 11, 2012 at 5:08 am** said:

Thanks

Tom
on **July 13, 2012 at 1:22 am** said:

Nice tutorial !
One clarification : For your data function, we have
function DataCost(i, label)
y = i / imageWidth
x = i – y*imageWidth
…….
Clearly, while calculating x, if we replace “y” with its definition from above, we get
x = i – i = 0
Was there a typo somewhere ??
Also, this is more of a conceptual question : as far as i knw, a disparity image is just the “difference” of the left and right images. Now, I am wondering how the disparity image is almost similar to the actual image
thanks !

nghiaho12 on **July 13, 2012 at 6:54 am** said:

The formula is correct, but I forgot to mention they’re integer operations. It should be

y = (int)(i / imageWidth)

For example, if we have an image that is 640×480, and look at the pixel at (200,100), then

i = 100*imageWidth + 200
i = 100*640 + 200
i = 64200

Using the equation to decompose back to (x,y) we get

y = (int)(64200/640) = 100 (rounded down)
x = 64200 – 10*640 = 200

which confirms it is correct.

The disparity image indicates the 'depth' of each pixel. If you were to manually do it by looking at each pixel in the left image and finding where it appears in the right image, find the absolute difference in x values, you'll get the disparity map. It's basically like a 3D image, that's why it looks somewhat similar to the original image.

Pingback: loopy belief propagation | Shirley's Blog

Moritz
on **April 10, 2013 at 10:30 am** said:

Hi,

nice description of LBP!

One minor issue though:
In section "Min-Sum – normalisation", the equation for A seems to be incorrect:
I think it should be A = – log(SUM(exp(– msg)))
That's the equivalent of "Sum-Product – normalisation" in log-space.

nghiaho12 on **April 11, 2013 at 4:53 am** said:

This is how I did it in my head:

Let M = [0.1 0.2 0.3], a message vector that needs to be normalised. Assume M is in normal probability space eg. [0,1], then

M_norm = M / sum(M) —> [0.16667 0.33333 0.50000]

Now let's see if we can derive the same result above in log space.

B = log(M) —> [-2.3026 -1.6094 -1.2040]
A = log(sum(exp(B))) —> -0.51083
B_norm = (B – A) —> [-1.79176 -1.09861 -0.69315]

Verifying the results.
exp(B_norm) —> [0.16667 0.33333 0.50000]

Same as M_norm.
which is the same as M_norm.

Did I do something wrong?

Viky
on **August 30, 2013 at 3:05 pm** said:

Hi,

Nice tutorial! It helps me a lot.

A small concern: for your data term.

function DataCost(i, label)
d = abs(leftImage(x,y) – rightImage(x – label),y)
end

I just do not quite understand why there is a ”-lable” in “rightImage(x – label,y)”,

Can you simply explain?

Many thanks

nghiaho12 on **August 30, 2013 at 6:53 pm** said:

Hi,

You can think of the variable “label” as “disparity in pixels”. This cost function was designed for the stereo image I used in this tutorial.
I’m basically doing, for a given pixel in the left image at (x,y) look at other possible pixels in the right image along

the x-axis, in the range [x - MAX_DISPARITY, x].
I'm exploiting the fact that a pixel in the left image cannot have a matching pixel greater than position 'x' in the right image.

> Viky on **August 30, 2013 at 9:02 pm** said:
>
> Thanks very much for your help. I will read your code and the tutorial carefully. It really helps.
> Thanks a lot for sharing.

Ricky
on **August 31, 2013 at 4:37 pm** said:

Hello,
very nice tutorial.
I was trying to implement the code in MATLAB.
You have not done any normalisation, so most of the values are reaching 255 after first iteration.
what should be the normalisation.
Should there be any data truncation for data cost.

> nghiaho12 on **August 31, 2013 at 6:38 pm** said:
>
> You shouldn't need any normalisation. If you're seeing 255 in the disparity map then something is wrong because I've set the limit to 16.

> Ricky on **September 1, 2013 at 12:57 am** said:
>
> Hello,
> Lets say the direction is Right,
> then the message mrf.grid(y,x).msg(2,label)
> increases as x increases in the right direction, the message finally reach 255.
> For finding the MAP assignment, cost should adds up to greater than 255,
> but since cost is uint8, it also remain 255.

Please help

> nghiaho12 on **September 1, 2013 at 1:08 am** said:
>
> This is possible. That's why you see in my code I use an unsigned int in the Pixel struct.

Viky
on **September 9, 2013 at 7:26 pm** said:

Hi Nghia,
I just noticed that, to find the minimum cost of assigning two labels (say label size is L) to two adjacent grids, we need to do O(L^2) message updates, right? It's computationally expensive.

I also noticed that someone has new methods like min-convolution that can reduce the complexity to o(L), are you familiar with that? I hope can talk with you further.

Sincerely,
Viky

> nghiaho12 on **September 10, 2013 at 6:39 am** said:
>
> Yep it's O(L^2), if you look in my code in the function SendMsg, there's 2 nested for loops over the number of labels (L). It's only expensive if you have an expensive cost function. For my stereo example the cost is extremely cheap, so the computation cost is irrelevant. But if you have an expensive cost function it might be possible to cache the results if you have enough memory.

nghiaho12 on **September 10, 2013 at 6:40 am** said:

I'm not familiar with the newer methods. I haven't put more time into belief propagation than what is already presented in my blog.

Viky
on **September 10, 2013 at 1:52 pm** said:

Thank you very much for your reply~

xu xu
on **February 9, 2014 at 3:44 am** said:

several formulas can not be seen in website, would you email a pdf version for me? thank you!